Summation and Limits

Sum of Constant

$$\sum_{i=1}^{n} c = cn$$

Sum with coefficient

$$\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$$

Sum of a Sum/Difference

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Sum of Linear Term

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Sum of Squared Term

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum of Cubed Term

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Limits to Infinity

$$\lim_{n\to\infty}\frac{1}{n} = 0 \qquad \lim_{n\to\infty}\frac{an^c}{bn^c} = \frac{a}{b}$$

Examples:

$$\sum_{i=1}^{8} i = \frac{8(9)}{2} = 36 \qquad \sum_{i=1}^{3} i^2 = \frac{3(4)(7)}{6} = 14 \qquad \sum_{i=1}^{5} i^3 = \frac{5^2 6^2}{4} = 225$$

Evaluate the sum:

1. $\sum_{i=1}^{5}\frac{i}{2} = \frac{1}{2}\left(\frac{5(6)}{2}\right) = \frac{15}{2} = 7.5$

2. $\sum_{i=1}^{5}\frac{3}{4}i^2 = \frac{3}{4}\left(\frac{5(6)(11)}{6}\right) = \frac{165}{4}$ or $41\frac{1}{4}$

Evaluate the sum S(n) in terms of n, then find $\lim_{n\to\infty} S(n)$:

3. $S(n) = \sum_{i=1}^{n}\frac{i}{n^2} = \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{n+1}{2n}$

$\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$

4. $S(n) = \sum_{i=1}^{n}\frac{i+2}{n^2} = \frac{1}{n^2}\left(\frac{n(n+1)}{2} + 2n\right)$

$= \frac{n+1}{2n} + \frac{2(2)}{n(2)} = \frac{n+5}{2n}$

$\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\frac{n+5}{2n} = \frac{1}{2}$

5. $S(n) = \sum_{i=1}^{n}\left[\left(1+\frac{i}{n}\right)^2\left(\frac{1}{n}\right)\right] = \frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right)$

$= \frac{1}{n}\left(n + \frac{\cancel{2}}{n}\left(\frac{n(n+1)}{\cancel{2}}\right) + \frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right)$

$\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(1 + \frac{n+1}{n} + \frac{(n+1)(2n+1)}{6n^2}\right)$

$= 1 + 1 + \frac{1}{3} = 2\frac{1}{3} = \frac{7}{3}$

6. $S(n) = \sum_{i=1}^{n}\left[3-2\left(\frac{i}{n}\right)\right]\left(\frac{1}{n}\right) = \frac{1}{n}\left(3n - \frac{\cancel{2}}{n}\frac{(n)(n+1)}{\cancel{2}}\right)$

$= 3 - \frac{n+1}{n}$

$\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(3 - \frac{n+1}{n}\right) = 3 - 1 = 2$

Parametric Equations

Curves are often represented by parametric equations, where a third variable (or parameter) is introduced to give more information about the curve. A common parameter might be t or θ . To convert a set of parametric equations into rectangular equations, solve for the parameter (t) in terms of one variable, substitute for t in the other equation. Rewrite the new equation, eliminating the t.

Example: $x = 2t - 3,\ y = 3t + 1$

Solve for t: $t = \dfrac{x+3}{2}$ **Substitute and eliminate t:** $y = 3\left(\dfrac{x+3}{2}\right) + 1 = \dfrac{3x}{2} + \dfrac{11}{2}$

You can now graph the parametric equation by graphing the new rectangular equation. Draw in direction arrows to indicate which way the curve (or line) is moving as t increases. For parametric equations involving trig functions, make use of identities like $\sin^2\theta + \cos^2\theta = 1$ to eliminate θ

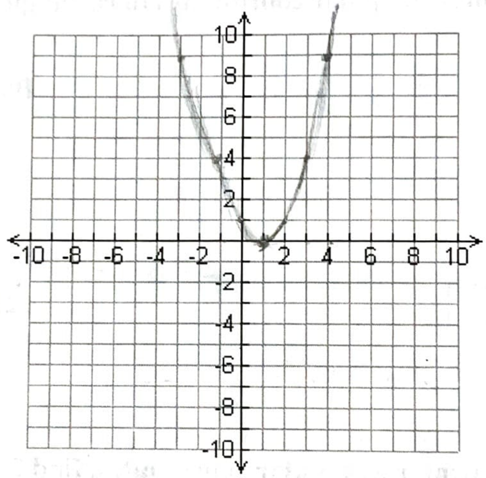
Rewrite the following parametric equations into rectangular equation. Then, sketch the curve and indicate the direction of the curve (via direction arrows).

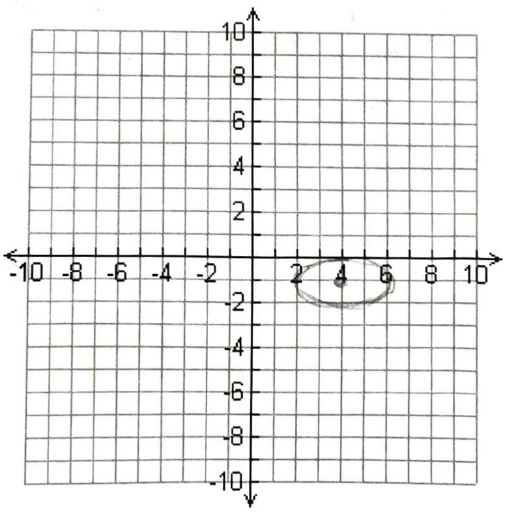
7. $x = t + 1 \Rightarrow t = x - 1$

$y = t^2 \Rightarrow y = (x-1)^2$

$y \geq 0$

parabola with vertex $(1, 0)$





8. $x = 4 + 2\cos\theta$

$y = -1 + \sin\theta$

$x - 4 = 2\cos\theta$

$\dfrac{x-4}{2} = \cos\theta$

$y + 1 = \sin\theta$

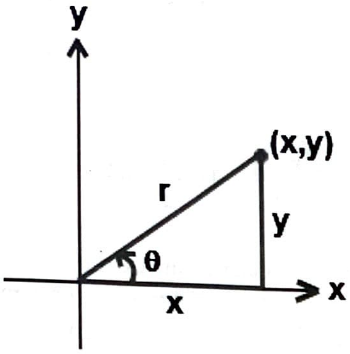
$(y+1)^2 + \left(\dfrac{x-4}{2}\right)^2 = 1$

$\dfrac{(x-4)^2}{2^2} + \dfrac{(y+1)^2}{1} = 1$

Ellipse with center $(4, -1)$ $a = 2$ $b = 1$

Polar Coordinates and Polar Equations

The polar coordinates (r, θ) are related to the rectangular coordinates (x,y) as follows. You should be able to deduce these relationships by examining the following right triangle.



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

$$\tan\theta = \frac{y}{x}$$

Convert the following polar coordinates to rectangular coordinates:

r, θ

9. $\left(8, \frac{\pi}{2}\right) = (0, 8)$

$x = 8\cos\frac{\pi}{2} = 0$

$y = 8\sin\frac{\pi}{2} = 8$

10. $\left(-2, \frac{5\pi}{3}\right) = (-1, \sqrt{3})$

$x = -2\cos\frac{5\pi}{3} = -2\left(\frac{1}{2}\right) = -1$

$y = -2\sin\frac{5\pi}{3} = -2\left(\frac{-\sqrt{3}}{2}\right) = \sqrt{3}$

11. $\left(-4, \frac{-3\pi}{4}\right) = (2\sqrt{2}, 2\sqrt{2})$

$x = -4\cos\left(\frac{5\pi}{4}\right) = -4\left(\frac{-\sqrt{2}}{2}\right) = 2\sqrt{2}$

$y = -4\sin\left(\frac{5\pi}{4}\right) = 2\sqrt{2}$

12. $\left(0, \frac{-7\pi}{6}\right) = (0, 0)$

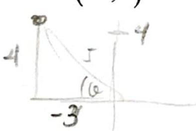
Given the following rectangular coordinates, find 2 sets of polar coordinates of the point for $0 \le \theta \le 2\pi$.

13. $(4,6) = (2\sqrt{13}, .983)$ or $(-2\sqrt{13}, .983 + \pi)$

$r^2 = 16 + 36 = 52$

$r = 2\sqrt{13}$

14. $(-3,4) = (5, 2.214)$ or $(-5, 2.214 + \pi)$



$\tan\theta = \frac{4}{3}$

$\theta = .927$

In Quad 2, $\theta = 2.214$

$r^2 = 9 + 16 = 25$

$r = 5$

15. $(-\sqrt{3}, -\sqrt{3}) = \left(\sqrt{6}, \frac{5\pi}{4}\right)$ or $\left(-\sqrt{6}, \frac{\pi}{4}\right)$

$r = \sqrt{3+3} = \sqrt{6}$

$\tan\theta = 1 \quad \theta = 45°$ or $\frac{\pi}{4}$

In Quad III, $\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

16. $(3,-1) = (\sqrt{10}, 5.961)$ or $(-\sqrt{10}, 2.820)$

$r = \sqrt{9+1} = \sqrt{10}$

$\tan\theta = \frac{1}{3}$

$\theta = .322$

In Quad IV

$\theta = 2\pi - .322$

or $\pi - .322$

Convert the rectangular equation to polar form:

17. $x^2+y^2=9$

$r^2 = 9$

$r = 3$

18. $3x-y+2=0$

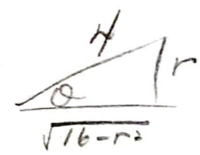
$3r\cos\theta - r\sin\theta + 2 = 0$

$r(3\cos\theta - \sin\theta) = -2$

19. $y^2-8x-16=0$

$r^2\sin^2\theta - 8r\cos\theta - 16 = 0$

Convert the polar equation to rectangular form:



20. $r=4\sin\theta$

$\sin\theta = \frac{r}{4}$

$r\sin\theta = \frac{r^2}{4}$

$y = \frac{x^2+y^2}{4}$

$4y = x^2+y^2$

$0 = y^2-4y+x^2$

$(y^2-4y+4)+x^2 = 4$

$\frac{(y-2)^2}{4} + \frac{x^2}{4} = 1$

circle centered at $(0,2)$
radius $= 2$

21. $\theta=\frac{\pi}{6}$

$\tan\frac{\pi}{6} = \frac{y}{x}$

$\frac{1/2}{\sqrt{3}/2} = \frac{y}{x}$

$\frac{y}{x} = \frac{1}{\sqrt{3}}$

$y = \frac{1}{\sqrt{3}}x$

22. $r=\frac{6}{2-3\sin\theta}$ $\left(\frac{1/2}{1/2}\right)$

$r = \frac{3}{1-\frac{3}{2}\sin\theta}$

$r = \frac{(3/2)2}{1-\frac{3}{2}\sin\theta}$

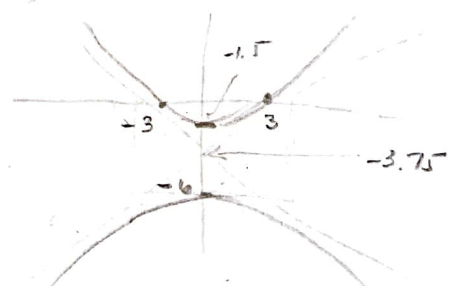
$e = 3/2 > 1$ hyperbola opens up

$p = 2$ distance from focus and directrix

$2r - 3y = 6$

$2\sqrt{x^2+y^2} - 3y = 6$

$e = \frac{c}{a} = \frac{3}{2}$



$\frac{3}{2} = \frac{c}{2.25}$

5 $a = 2.25$

$b =$

$3.375 = c$

$b = 2.5156$

Sketch the graph of the polar equation by hand, i.e. without using a graphing calculator.

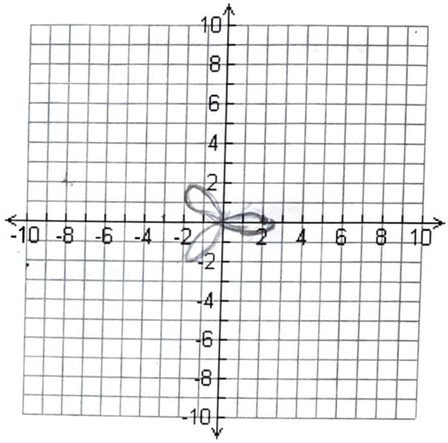
NOTE: You can use the calculator afterwards to verify that your graph is correct, but I do expect you to be able to sketch these graphs by hand (i.e. you may be asked to sketch polar graphs on a no calculator quiz).

23. $r = 2\cos(3\theta)$

$3\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{6}$

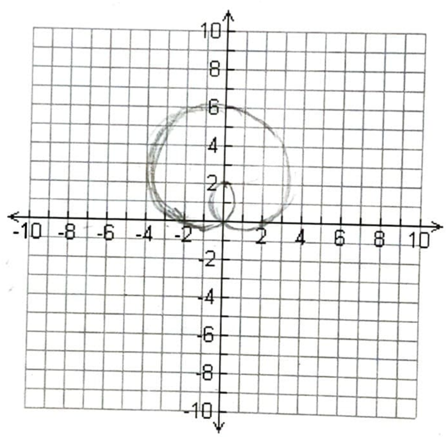
θ	r
0	2
$\frac{\pi}{6}$	0
$\frac{\pi}{3}$	-2
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	2
$\frac{5\pi}{6}$	0
π	-2



24. $r = 2 + 4\sin\theta$

$r_1 = 4\sin\theta$

θ	r_1	r
0	0	2
$\frac{\pi}{2}$	4	6
π	0	2
$\frac{3\pi}{2}$	-4	-2
2π	0	2



25. $r = 3(1 - \cos\theta)$

θ	r
0	0
$\frac{\pi}{2}$	3
π	6
$\frac{3\pi}{2}$	3
2π	0